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ELASTIC EQUILIBRIUM OF A MEDIUM CONTAINING A FINITE NUMBER OF ALIGNED SPHEROIDAL INCLUSIONS

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Abstract—The strict solution in series is obtained of the elasticity theory problem for an unbounded domain containing some aligned spheroidal inhomogeneities under uniform far-field loads. The essence of the method used is the representation of the displacement field in a multiply-connected domain as a sum of general solutions for corresponding single-connected domains. Each term of this sum, in turn, is expanded into series on vectorial partial solutions of Lame's equation in a local spheroidal basis. In order to satisfy exactly all interfacial boundary conditions, the re-expansion formulae (addition theorems) for external partial solutions are used. As a result, the primary boundary-value problem of elasticity theory is reduced to an infinite set of linear algebraic equations. The convergence rate of the proposed solution procedure is evaluated numerically. Some numerical results demonstrating the influence on stress distribution of material properties, spatial position of inclusions and external load are presented.

1. INTRODUCTION

The modern advanced particle composites with superior mechanical properties are, as a rule, strongly heterogeneous materials with high volume content of the dispersed phase. The global behaviour of these composites is strongly influenced by the spatial distribution and interaction among the microconstituents. To estimate this interaction (and, therefore, material properties) accurately, a model which effectively represents the composite microstructure and a rigorous method to analyse corresponding model problems are required. The appropriate model of such composites is the infinite region containing some inhomogeneities. In order to determine the response of this model the boundary-value problem for a multiply-connected domain must be solved.

The number of publications where three-dimensional problems for a multiply-connected body were considered is rather limited. For the case of spherical inhomogeneities there are some papers containing the rigorous solutions. The axisymmetrical two-sphere problems were treated in terms of bispherical coordinates by Sternberg and Sadowsky (1952) for cavities and by Shelley and Yu (1966) for rigid inclusions. The solution for elastic spheres was obtained by Chen and Acrivos (1978). Their analysis is based on the Boussinesq–Papkovich stress function approach and makes use of the "multipole expansion" technique in which the solutions are expanded into series of spherical harmonics with respect to the centres of spheres. The same approach was used by Tsuchida *et al.* (1976), who solved the problem when the applied field consists of a uniaxial tension in the direction perpendicular to the line of centres of the cavities. A more general mathematical technique was applied by Golovchan (1974) to the problem for a space with N arbitrary placed non-touching spherical inclusions under a general type of loading. The essence of the method proposed is the representation of the matrix displacement vector as a sum of general solutions for a single-particle problem. The latter are expressed by series of partial vectorial solutions of Lame's equation in a spherical basis. The full satisfaction of interfacial boundary conditions is achieved by using the re-expansion formulae for external partial solutions in a transferred coordinate system (addition theorems) derived by Golovchan (1974). This procedure leads to an infinite linear algebraic system with normal determinant. Unfortunately, the numerical results confirming the computational effectiveness of the method are not presented.

For a more general (ellipsoidal) shape of inhomogeneities only approximate solutions of the multi-particle problem are known, which are based on Eshelby's solution for a single inclusion in an unbounded domain (Eshelby, 1959). The method proposed by Robin and Hwang (1991) uses two assumptions: the original problem for N inhomogeneities can be represented by N separate problems for a single inhomogeneity; the contributions of the remaining inhomogeneities to the far field of a reference inhomogeneity are based only on their average equivalent transformation strains. In the solution obtained in this way the interfacial conditions are satisfied only approximately. Another approach to this problem was developed by Hori and Nemat-Nasser (1993). In a proposed "double-inclusion" model the average field quantities are estimated with the aid of a theorem generalizing the Mori–Tanaka theory.

A more rigorous approach to the problem for a medium containing two ellipsoidal inhomogeneities has been suggested by Moschovidis and Mura (1975). The fundamental idea of the method used is the representation of transformation strain within each domain by a polynomial in Cartesian coordinates. The decomposition of the strain field around inhomogeneities into a Taylor series reduces the problem to an algebraic system. The expressions of matrix coefficients include the partial derivatives from potential functions of complex form. The accuracy of the solution obtained depends on the highest polynomial degree of the Taylor expansion. Its increase is connected with the calculation of higher derivatives and leads to a considerable complication of the expressions for matrix coefficients. It is difficult to estimate the computational effectiveness of the method because the numerical results are presented only for far removed inclusions.

The solution of the elasticity problem stated below for a space with a finite number of aligned spheroidal inclusions is obtained using a method similar to that outlined by Golovchan (1974). The necessary mathematical results, namely the vectorial partial solutions of Lame's equations in a spheroidal basis and the addition theorems for external partial solutions, are given in Appendices A and B, respectively.

2. MEDIUM WITH A SINGLE INCLUSION

We first consider the simplest problem for an unbounded domain with a single spheroidal cavity. We introduce the Cartesian coordinate system (x, y, z) with an origin in the centre of the spheroid and the corresponding spheroidal coordinates (f, ξ, η, φ) (A3) so that the surface of the cavity coincides with the coordinate surface $\xi = \xi^0$. The stresses in a medium with a cavity are induced by the applied uniform symmetrical remote strain tensor ε^∞. The surface of the cavity is supposed to be stress-free. We present the displacement vector in the region $\xi \geqslant \xi^0$ in the form

$$\mathbf{u}^{(0)} = \varepsilon^\infty \cdot \mathbf{r} + \mathbf{U}^{(1)}, \tag{1}$$

where $\mathbf{r}$ is the radius vector and $\mathbf{U}^{(1)}$ is the disturbance caused by the presence of a cavity. Because for $\|\mathbf{r}\| \to \infty$ $\mathbf{U}^{(1)} \to 0$, this series development contains the external solutions (A4) only:

$$\mathbf{u}_1 = \sum_{i=1}^{3} \sum_{t=0}^{\infty} \sum_{s=-t}^{t} A_{ts}^{(i)} \mathbf{S}_{ts}^{(i)}(\mathbf{r}, f), \tag{2}$$

where $A_{ts}^{(i)}$ are the constants to be determined.

The expansion of the linear part of eqn (1), on the contrary, contains only internal solutions (A1). Taking into account the form of the partial solutions

$$f\mathbf{s}_{20}^{(1)} = -\tfrac{1}{2}(x\mathbf{e}_x + y\mathbf{e}_y) + z\mathbf{e}_z\,; \quad f(\mathbf{s}_{21}^{(1)} + \mathbf{s}_{2,-1}^{(1)}) = i(z\mathbf{e}_y - y\mathbf{e}_z)\,;$$

$$f(\mathbf{s}_{21}^{(1)} - \mathbf{s}_{2,-1}^{(1)}) = z\mathbf{e}_x + x\mathbf{e}_z\,; \quad 2f(\mathbf{s}_{22}^{(1)} + \mathbf{s}_{2,-2}^{(1)}) = x\mathbf{e}_x - y\mathbf{e}_y\,;$$

$$2f(\mathbf{s}_{22}^{(1)} - \mathbf{s}_{2,-2}^{(1)}) = i(x\mathbf{e}_y + y\mathbf{e}_x); \quad 2f\mathbf{s}_{10}^{(2)} = i(y\mathbf{e}_x - x\mathbf{e}_y);$$
$$f(\mathbf{s}_{11}^{(2)} + \mathbf{s}_{1,-1}^{(2)}) = z\mathbf{e}_x; \quad f(\mathbf{s}_{11}^{(2)} - \mathbf{s}_{1,-1}^{(2)}) = iz\mathbf{e}_y;$$
$$f\mathbf{s}_{00}^{(3)} = \tfrac{2}{3}(2\nu-1)(x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z) = \tfrac{2}{3}(2\nu-1)\mathbf{r}, \tag{3}$$

we find after transformations

$$\mathbf{u}^{(0)} = \sum_{i=1}^{3} \sum_{t=0}^{\infty} \sum_{s=-t}^{t} [A_{ts}^{(i)}\mathbf{S}_{ts}^{(i)}(\mathbf{r},f) + b_{ts}^{(i)}\mathbf{s}_{ts}^{(i)}(\mathbf{r},f)], \tag{4}$$

where

$$b_{00}^{(3)} = \frac{f}{(2\nu-1)}(\varepsilon_{11}^{\infty} + \varepsilon_{22}^{\infty} + \varepsilon_{33}^{\infty}); \quad b_{20}^{(1)} = \frac{f}{3}(2\varepsilon_{33}^{\infty} - \varepsilon_{11}^{\infty} - \varepsilon_{22}^{\infty});$$
$$b_{21}^{(1)} = \overline{b_{2,-1}^{(1)}} = f(\varepsilon_{13}^{\infty} - i\varepsilon_{23}^{\infty}); \quad b_{22}^{(1)} = \overline{b_{2,-2}^{(1)}} = f(\varepsilon_{11}^{\infty} - \varepsilon_{22}^{\infty} - 2i\varepsilon_{12}^{\infty});$$

all other $b_{ts}^{(i)}$ are equal to zero.

According to the problem statement, the stress vector

$$\mathbf{T}_{\xi} = \sigma_{\xi}\mathbf{e}_{\xi} + \tau_{\xi\eta}\mathbf{e}_{\eta} + \tau_{\xi\varphi}\mathbf{e}_{\varphi} = 2\mu\left(\frac{\nu}{(1-2\nu)}\mathbf{e}_{\xi}\nabla\cdot\mathbf{u} + \frac{\xi\eta}{f}\frac{\partial}{\partial\xi}\mathbf{u} + \tfrac{1}{2}\mathbf{e}_{\xi}\times\nabla\times\mathbf{u}\right) \tag{5}$$

on the surface $\xi = \xi^0$ is equal to zero. In eqn (5), ν is Poisson's ratio, μ is the shear modulus, $\mathbf{e}_{\xi} = n_i\mathbf{e}_i$, $n_1 = \bar{n}_2 = h\xi\bar{\eta}e^{-i\varphi}$, $n_3 = h\bar{\xi}\eta$ and $h = (\xi^2 - \eta^2)^{-1/2}$. Taking into account eqn (4), this condition can be written as

$$\mathbf{T}_{\xi}(\mathbf{u}^{(0)})|_{\xi=\xi^0} = \sum_{i=1}^{3} \sum_{t=0}^{\infty} \sum_{s=-t}^{t} [A_{ts}^{(i)}\mathbf{T}_{\xi}(\mathbf{S}_{ts}^{(i)}) + b_{ts}^{(i)}\mathbf{T}_{\xi}(\mathbf{s}_{ts}^{(i)})]. \tag{6}$$

Thus, we need to calculate eqn (5) for partial solutions. This is fairly straightforward if we use the properties of the vectorial functions $\mathbf{s}_{ts}^{(i)}$ and $\mathbf{s}_{ts}^{(i)}$ with respect to differential operators (Kushch, 1995). So, for external solutions we find

$$\gamma\mathbf{T}_{\xi}(\mathbf{S}_{ts}^{(1)}) = \mathbf{e}_1\frac{(t-s+2)!}{(t+s)!}Q_{t+1}^{\prime s-1}\chi_{t+1}^{s-1} - \mathbf{e}_2\frac{(t-s)!}{(t+s+2)!}Q_{t+1}^{\prime s+1}\chi_{t+1}^{s+1} + \mathbf{e}_3\frac{(t-s+1)!}{(t+s+1)!}Q_{t+1}^{\prime s}\chi_{t+1}^{s};$$

$$\gamma\mathbf{T}_{\xi}(\mathbf{S}_{ts}^{(2)}) = \mathbf{e}_1\frac{(t-s+1)!}{(t+s-1)!}\chi_t^{s-1}\left[\frac{(t+s)}{t}Q_t^{\prime s-1} - \frac{Q_t^s}{2\bar{\xi}}\right] + \mathbf{e}_2\frac{(t-s-1)!}{(t+s+1)!}\chi_t^{s+1}$$
$$\cdot(t-s)\left[\frac{Q_t^{\prime s+1}}{t} - \frac{(t+s+1)}{2\bar{\xi}}Q_t^s\right] + \mathbf{e}_3\frac{(t-s)!}{(t+s)!}\chi_t^s s\left[\frac{Q_t^{\prime s}}{t} + \frac{\xi}{2\bar{\xi}^2}Q_t^s\right];$$

$$\gamma\mathbf{T}_{\xi}(\mathbf{S}_{ts}^{(3)}) = \mathbf{e}_1\frac{(t-s)!}{(t+s-2)!}\chi_{t-1}^{s-1}\left\{\frac{2[\nu + s(\nu-1)/t]}{\bar{\xi}}Q_{t-1}^s\right.$$
$$\left. + (t-s+1)(Q_t^{s-1} - \xi Q_t^{\prime s-1}) + (t+s-1)[1 + (t+s)\beta_{-(t+1)}]Q_{t-1}^{\prime s-1}\right\}$$
$$-\mathbf{e}_2\frac{(t-s-2)!}{(t+s)!}\chi_{t-1}^{s+1}(t-s-1)\left\{\frac{2[\nu - s(\nu-1)/t]}{\bar{\xi}}(t+s)Q_{t-1}^s\right.$$

$$+Q_t^{s+1}-\xi Q_t'^{s+1}+[1+(t-s)\beta_{-(t+1)}]Q_{t-1}'^{s+1}\Big\}$$

$$+\mathbf{e}_3\frac{(t-s-1)!}{(t+s-1)!}\chi_{t-1}^s\left\{\frac{2(1-\nu)\xi s^2}{t\bar{\xi}^2}Q_{t-1}^s+(t-s)\left(Q_t^s-\xi Q_t'^s\right)\right.$$

$$\left.+[2(1-\nu)-C_{-(t+1),s}]Q_{t-1}'^s\right\},\quad \gamma=f/(2\mu_0\bar{\xi}h); \tag{7}$$

where the prime stands for differentiation with respect to the argument. For internal solutions the formulae are quite similar. By projection of eqn (6) on the orthogonal unit vectors $\mathbf{e}_i$ we obtain the scalar equalities

$$\sum_{i=1}^{3}\sum_{t=0}^{\infty}\sum_{s=-t}^{t}[A_{ts}^{(i)}T_{ts}^{(i)(j)}+b_{ts}^{(i)}t_{ts}^{(i)(j)}]=0,\quad j=1,2,3, \tag{8}$$

where

$$\mathbf{T}_\xi(\mathbf{S}_{ts}^{(i)})=T_{ts}^{(i)(j)}\mathbf{e}_j,\quad \mathbf{T}_\xi(\mathbf{s}_{ts}^{(i)})=t_{ts}^{(i)(j)}\mathbf{e}_j;$$

the form of expressions $T_{ts}^{(i)(j)}$ and $t_{ts}^{(i)(j)}$ is clear from eqns (7). Finally we decompose each of the equalities (8) over a full and orthogonal system of scalar harmonics χ_t^s. This gives us a set of linear algebraic equations with unknowns $A_{ts}^{(i)}$:

$$A_{t-1,s}^{(1)}Q_t'^s+A_{ts}^{(2)}s\left[\frac{Q_t'^s}{t}+\frac{\xi}{2\bar{\xi}^2}Q_t^s\right]+A_{t+1,s}^{(3)}\left\{\frac{2(1-\nu)s^2\xi}{(t+1)\bar{\xi}^2}Q_t^s\right.$$

$$\left.+(t-s+1)(Q_{t+1}^s-\xi Q_{t+1}'^s)+[2(1-\nu)-C_{-(t+2),s}]Q_t'^s\right\}$$

$$+b_{t+1,s}^{(1)}P_t'^s+b_{ts}^{(2)}s\left[-\frac{P_t'^s}{t+1}+\frac{\xi}{2\bar{\xi}^2}P_t^s\right]+b_{t-1,s}^{(3)}\left\{-\frac{2(1-\nu)s^2\xi}{t\bar{\xi}^2}P_t^s\right.$$

$$\left.-(t+s)(P_{t-1}^s-\xi P_{t-1}'^s)+[2(1-\nu)-C_{t-1,s}]P_t'^s\right\}=0,$$

$$t=1,2,\ldots,\quad |s|\leqslant t;$$

$$A_{t-1,s}^{(1)}Q_t'^{s-1}+A_{ts}^{(2)}\left[\frac{t+s}{t}Q_t'^{s-1}+\frac{Q_t^s}{2\bar{\xi}}\right]+A_{t+1,s}^{(3)}\left\{\frac{2[\nu(t+s+1)-s]}{(t+1)\bar{\xi}}\right.$$

$$\left.\cdot Q_t^s+(t-s+2)(Q_{t+1}^{s-1}-\xi Q_{t+1}'^{s-1})+(t+s)[1+(t+s+1)\beta_{-(t+2)}]Q_t'^{s-1}\right\}$$

$$+b_{t+1,s}^{(1)}P_t'^{s-1}+b_{ts}^{(2)}\left[\frac{t-s+1}{t+1}P_t'^{s-1}-\frac{P_t^s}{2\bar{\xi}}\right]+b_{t-1,s}^{(3)}\left\{\frac{2[\nu(t-s)+s]}{t\bar{\xi}}P_t^s\right.$$

$$\left.-(t+s-1)(P_{t-1}^{s-1}-\xi P_{t-1}'^{s-1})+(t-s+1)[-1+(t-s)\beta_{t-1}]P_t'^{s-1}\right\}=0,$$

$$t=1,2,\ldots,\quad |s-1|\leqslant t;$$

$$A_{t-1,s}^{(1)}Q_t'^{s+1}-A_{ts}^{(2)}(t-s)\left[\frac{1}{t}Q_t'^{s+1}-\frac{(t+s+1)}{2\bar{\xi}}Q_t^s\right]+A_{t+1,s}^{(3)}(t-s)$$

$$\cdot\left\{\frac{2[\nu(t-s+1)+s]}{(t+1)\xi}(t+s+1)Q_t^s+Q_{t+1}^{s+1}-\xi Q_{t+1}^{\prime s+1}\right.$$

$$\left.+[1+(t+s+1)\beta_{-(t+2)}]Q_t^{\prime s+1}\right\}+b_{t+1,s}^{(1)}P_t^{\prime s+1}-b_{ts}^{(2)}(t+s+1)$$

$$\cdot\left[\frac{P_t^{\prime s+1}}{t+1}-\frac{t-s}{2\xi}P_t^s\right]+b_{t-1,s}^{(3)}(t+s+1)\left\{\frac{2[\nu(t+s)-s]}{t\xi}(t-s)P_t^s\right.$$

$$\left.-P_{t-1}^{s+1}+\xi P_{t-1}^{\prime s+1}+[-1+(t+s)\beta_{t-1}]P_t^{\prime s+1}\right\}=0,$$

$$t=1,2,\ldots,\quad |s+1|\leqslant t. \tag{9}$$

Equations (9) can be rewritten in a matrix mode:

$$TG_t(\nu)\mathbf{A}_t+TM_t(\nu)\mathbf{b}_t=0,\quad t=1,2,\ldots, \tag{10}$$

where the vector $\mathbf{A}_t$ includes unknowns $A_{t+i-2,s}^{(i)}$ and the vector $\mathbf{b}_t$ includes values $b_{t+2-i,s}^{(i)}$. The structure of matrices TG_t and TM_t is clear from eqns (9). It follows from eqn (4) that $\mathbf{b}_t$ are non-zero only for $t=1$. Then, from eqn (10) also $\mathbf{A}_t=0$ for $t\geqslant 2$. Thus, the problem is reduced to the linear system (9) for $t=1$, containing six equations. For the parameters $\varepsilon_{ij}^{\infty}$ given the system (9), and the problem examined therefore has only one solution.

The problem for the space with the spheroidal inhomogeneity is considered in the same manner. The only difference is the interfacial boundary conditions, where displacements and normal stresses are supposed to be continuous:

$$[\mathbf{u}^{(0)}-\mathbf{u}^{(1)}]_{\xi=\xi^0}=0;\quad [\mathbf{T}_\xi(\mathbf{u}^{(0)})-\mathbf{T}_\xi(\mathbf{u}^{(1)})]_{\xi=\xi^0}=0, \tag{11}$$

where $\mathbf{u}^{(1)}$ is the displacement vector within inclusion. Because $\mathbf{u}^{(1)}$ is limited in the volume of inclusion, this series expansion consists only of internal partial solutions:

$$\mathbf{u}^{(1)}=\sum_{i=1}^{3}\sum_{t=0}^{\infty}\sum_{s=-t}^{t}D_{ts}^{(i)}\mathbf{s}_{ts}^{(i)}(\mathbf{r},f), \tag{12}$$

where $D_{ts}^{(i)}$ are the indefinite constants. The substitution of eqns (4) and (12) into eqn (11) leads us to the system of equations

$$\begin{aligned}UG_t(\nu_0)\mathbf{A}_t+UM_t(\nu_0)\mathbf{b}_t&=UM_t(\nu_1)\mathbf{D}_t,\\ TG_t(\nu_0)\mathbf{A}_t+TM_t(\nu_0)\mathbf{b}_t&=æ TM_t(\nu_1)\mathbf{D}_t,\end{aligned} \tag{13}$$

where $æ=\mu_1/\mu_0$, $\mu=\mu_0$, $\nu=\nu_0$ for the matrix, $\mu=\mu_1$, $\nu=\nu_1$ for the inclusion. The structure of the matrices UM_t and UG_t in eqn (13) is the same as that of TM_t and TG_t. They are obtained by decomposition of the first condition (11) over $\mathbf{e}_i$ and χ_t^s. Taking into account (A5), this is not a problem.

By way of some matrix operations, the number of unknowns can be reduced twice. So, on the one hand, by elimination of $\mathbf{D}_t$ we find

$$[æ UM_t^{-1}(\nu_1)UG_t(\nu_0)-TM_t^{-1}(\nu_1)TG_t(\nu_0)]\mathbf{A}_t$$
$$+[æ UM_t^{-1}(\nu_1)UG_t(\nu_0)-TM_t^{-1}(\nu_1)TG_t(\nu_0)]\mathbf{b}_t=0. \tag{14}$$

On the other hand, the vectors $\mathbf{D}_t$ and $\mathbf{A}_t$ are connected by the relation

$$\mathbf{D}_t = [UM_t^{-1}(v_0)UM_t(v_1) - æTM_t^{-1}(v_0)TM_t(v_1)]^{-1} \cdot [UM_t^{-1}(v_0)UG_t(v_0) - TM_t^{-1}(v_0)TG_t(v_0)]\mathbf{A}_t. \quad (15)$$

Since $\mathbf{b}_t = 0$ for $t \neq 0$, then, as in the above case, $\mathbf{A}_t$ and $\mathbf{D}_t$ are non-zero only for $t = 1$.

The solution of the problem considered in the form (14), (15) from a mathematical standpoint is fully equivalent to that obtained previously (Eshelby, 1959; Podilchuk, 1967). The only differences are the use of vectorial partial solutions of Lame's equation and a matrix form for resolving the algebraic system.

3. MEDIUM WITH N PORES OR INCLUSIONS

We now consider an unbounded domain containing N non-touching aligned spheroidal cavities with centres O_q, defined by parameters $f = f_q$, $\xi = \xi_q^0$, $q = 1, N$. As before, the stressed state is induced by tensor ε^∞. The surface of all cavities is supposed to be stress free. When the elasticity theory problem for a multiply-connected body is under consideration, the question of an appropriate form of solution arises. This question has been studied in detail by Slobodyanskyj (1954), who has shown that this solution can be found as a sum of general solutions for corresponding single-connected domains. Therefore, we have

$$\mathbf{u}^{(0)} = \varepsilon^\infty \cdot \mathbf{r} + \sum_{q=1}^{N} \mathbf{U}^{(q)}, \quad (16)$$

where $\mathbf{U}^{(q)} = \mathbf{U}(\mathbf{r}_q, f_q)$ is the external solution of the form (2) for the infinite region external with respect to the surface $\xi = \xi_q^0$ and $\mathbf{r}_q$ is the radius vector of the local coordinate system $O_q x_q y_q z_q$. This system is introduced so that the $O_q z_q$ axis coincides with the rotaton axis of the qth spheroid. Thus, the problem consists of determination of unknown constants $A_{ts}^{(i)(q)}$ in the expression of the matrix displacement vector $\mathbf{u}^{(0)}$:

$$\mathbf{u}^{(0)} = \varepsilon^\infty \cdot \mathbf{r}_1 + \sum_{q=1}^{N} \sum_{i=1}^{3} \sum_{t=0}^{\infty} \sum_{s=-t}^{t} A_{ts}^{(i)(q)} \mathbf{S}_{ts}^{(i)}(\mathbf{r}_q, f_q) \quad (17)$$

from boundary conditions

$$\mathbf{T}_{\xi_q}(\mathbf{u}^{(0)})|_{\xi_q = \xi_q^0} = 0, \quad q = 1, 2, \ldots, N. \quad (18)$$

Because these conditions are written in a local coordinate system, the displacement vector and the corresponding stress vector must be transformed to this local basis. To complete such transformation we will use the addition theorems for external solutions of Lame's equation (Appendix B). Considering that $\mathbf{r}_q = \mathbf{R}_{qn} + \mathbf{r}_n$, where $\mathbf{R}_{qn}$ is the vector connecting points O_q and O_n, we obtain, after change of summation order,

$$\mathbf{u}^{(0)} = \varepsilon^\infty \cdot \mathbf{R}_{1n} + \sum_{i=1}^{3} \sum_{t=0}^{\infty} \sum_{s=-t}^{t} [A_{ts}^{(i)(n)} \mathbf{S}_{ts}^{(i)}(\mathbf{r}_n, f_n) + (a_{ts}^{(i)(n)} + b_{ts}^{(i)}) \mathbf{s}_{ts}^{(i)}(\mathbf{r}_n, f_n)], \quad (19)$$

where

$$a_{ts}^{(i)(n)} = \sum_{j=1}^{3} \sum_{k=0}^{\infty} \sum_{l=-k}^{k} \sum_{q=1}^{N}{}' \eta_{ktls}^{(j)(i)}(\mathbf{R}_{qn}, f_q, f_n) A_{kl}^{(j)(q)}. \quad (20)$$

The stroke over the internal sum sign denotes the absence of a term with $q = n$.

The constant vector $\varepsilon^\infty \cdot \mathbf{R}_{1n}$ determining the transfer of a whole solid does not contribute to the total stress tensor. For other terms, by substitution of eqn (19) into eqn (18) we obtain the expression

$$\mathbf{T}_{\xi_n}(\mathbf{u}^{(0)})|_{\xi_n=\xi_n^0} = \sum_{i=1}^{3}\sum_{t=0}^{\infty}\sum_{s=-t}^{t}\{A_{ts}^{(i)(n)}\mathbf{T}_{\xi_n}[\mathbf{S}_{ts}^{(i)}(\mathbf{r}_n,f_n)] + (a_{ts}^{(i)(n)}+b_{ts}^{(i)})\mathbf{T}_{\xi_n}[\mathbf{S}_{ts}^{(i)}(\mathbf{r}_n,f_n)]\}|_{\xi_n=\xi_n^0} = 0, \tag{21}$$

similar to eqn (6). The application of the standard transition procedure stated above gives us the set of algebraic equations

$$TG_t^{(n)}(v_0)\mathbf{A}_t^{(n)} + TM_t^{(n)}(v_0)(\mathbf{b}_t + \mathbf{a}_t^{(n)}) = 0, \quad t = 1,2,\ldots, \tag{22}$$

where the vector $\mathbf{a}_t^{(n)}$ has the same structure as $\mathbf{b}_t$ and contains terms $\mathbf{a}_{t+2-i,s}^{(i)(n)}$. The vector $\mathbf{a}_t^{(n)}$ can be rewritten in the form

$$\mathbf{a}_t^{(n)} = \sum_{k=0}^{\infty}\sum_{q=1}^{N}{}' \eta_{kt}(\mathbf{R}_{qn},f_q,f_n)\mathbf{A}_k^{(q)}; \tag{23}$$

this is merely a matrix form of eqn (20). By substitution of eqn (23) into eqn (22) we obtain an infinite system of linear algebraic equations with unknowns $\mathbf{A}_k^{(q)}$:

$$\mathbf{A}_t^{(n)} + [TG_t^{(n)}]^{-1}TM_t^{(n)}\sum_{k=0}^{\infty}\sum_{q=1}^{N}{}' \eta_{kt}(\mathbf{R}_{qn},f_q,f_n)\mathbf{A}_k^{(q)} = -[TG_t^{(n)}]^{-1}TM_t^{(n)}\mathbf{b}_t; \quad t = 1,2,\ldots; \quad q = 1,2,\ldots,N. \tag{24}$$

The analysis of coefficients of matrices $TG_t^{(n)}$, $TM_t^{(n)}$ and η_{kt} shows that eqn (24) is a system with normal determinant. Hence, the solution of eqn (24) can be obtained by either a reduction method or a method of successive approximations. The exactness of the solution obtained is defined by the maximum value of the index t retained in eqns (17)–(24). Note that with the number of cavities increased, the dimension of system (24) increases proportionally. Therefore, from the standpoint of rational arrangement of computations, it is advisable to use the iterative procedure with preliminary calculation of matrices $TG_t^{(n)}$, $TM_t^{(n)}$ and η_{kt} for solving eqn (24). The calculation of matrix coefficients is the most time-consuming part of the computational algorithm, but it is carried out only once. The following course of calculations consists of the refinement of values $\mathbf{A}_k^{(q)}$ by way of consequent satisfaction of boundary conditions on each of the cavities:

$$\mathbf{A}_{t,i+1}^{(n)} = -[TG_t^{(n)}]^{-1}TM_t^{(n)}(\mathbf{a}_{t,i}^{(n)}+\mathbf{b}_t), \quad i = 0,1,2,\ldots, \tag{25}$$

where $\mathbf{A}_{t,i}^{(n)}$ is the value of the vector $\mathbf{A}_t^{(n)}$ on the ith iteration, $\mathbf{A}_{t,0}^{(n)}$ is the solution of the problem for a space with a single cavity [$\mathbf{a}_{t,0}^{(n)} = 0$ in eqn (25)]. The results of calculation (in particular, presented below) show that the iterative procedure (23)–(25) converges rapidly enough. Even for nearly-touching cavities when $\|\mathbf{R}_{12}\| = 1.05(f_1\xi_1^0 + f_2\xi_2^0)$ the number of iterations $I_{\max}$ necessary for calculation of $A_{ts}^{(i)(n)}$ with relative error $\varepsilon = 10^{-4}$ does not exceed 20.

The stress determination in a medium with N spheroidal inclusions under conditions of full mechanical contact on interfaces,

$$[\mathbf{u}^{(0)} - \mathbf{u}^{(q)}]_{\xi_q=\xi_q^0} = 0; \quad [\mathbf{T}_{\xi_q}(\mathbf{u}^{(0)}) - \mathbf{T}_{\xi_q}(\mathbf{u}^{(q)})]_{\xi_q=\xi_q^0} = 0; \quad q = 1,2,\ldots,N, \tag{26}$$

where

$$\mathbf{u}^{(q)} = \sum_{i=1}^{3} \sum_{t=0}^{\infty} \sum_{s=-t}^{t} D_{ts}^{(i)(q)} \mathbf{s}_{ts}^{(i)}(\mathbf{r}_q, f_q) \tag{27}$$

is the displacement vector within the qth inclusion and $D_{ts}^{(i)(q)}$ are the indefinite constants, is carried out in the same way. In order to satisfy the boundary conditions one must transform eqn (19) to the qth local spheroidal basis and then substitute together with eqn (27) in eqn (26). The following decomposition of vectorial equalities obtained over $\mathbf{e}_i$ by χ_t^s gives us N systems of form (14), where $\mathbf{A}_t$ must be replaced on $\mathbf{A}_t^{(n)}$ and $\mathbf{b}_t$ on $\mathbf{b}_t + \mathbf{a}_t^{(n)}$. To solve this algebraic system, the iterative procedure (25) can also be used. Note that resolving the algebraic system obtained by Rodin and Hwang (1991) has a dimension $6N$; this corresponds to only unknowns with $t = 1$ being kept in the above stated solution.

4. NUMERICAL RESULTS

The model considered has many parameters. They are the number of inclusions, their size, shape, properties and spatial position, and external load. A complete parametric study of this model is not the subject of this article. We restrict our analysis to only the few simplest examples providing, however, the possibility of estimating the computational effectiveness of the proposed method. They also allow us to establish some typical peculiarities of the stress distribution caused by interaction of neighbouring inhomogeneities. We consider an unbounded elastic medium containing two equal spheroidal inhomogeneities $f_1 = f_2$, $\xi_1^0 = \xi_2^0$ with centres on axis Ox: $Y_{12} = Z_{12} = 0$ (Fig. 1). We put $\mu_0 = 1$, $\nu_0 = 0.3$ and $\nu_1 = \nu_2 = 0.2$. The aspect ratio of spheroids is $p_z/p_x = \xi_2^0/\bar{\xi}_2^0 = 2$, where $p_x = f\bar{\xi}$ and $p_z = f\xi$ are the semi-axes of the spheroid. The variable parameters are the shear moduli of inclusions μ_1 and μ_2, the distance between centres $\|\mathbf{R}_{12}\| = X_{12}$ and the far-field load tensor. Note that it is an essentially three-dimensional problem for any loading type.

It is of interest to investigate the convergence of solution (17)–(24) when the maximum value $t = t_{max}$ increases as well as the convergence of iterative procedure (25), because this allows us to estimate the degree of accuracy of numerical results presented here. So, in Table 1 the values $A_{00}^{(1)(1)}$ as a function of t_{max} and i_{max} are presented. They are calculated for parameters $\mu_1 = \mu_2 = 0$, $X_{12} = 2.1p_x$, $\varepsilon_{33}^{\infty} = \delta$ (uniaxial deformation along the Oz axis). It is seen that even for the case of nearly placed cavities the convergence rate is rapid enough: for $t_{max} \geqslant 7$ the first four valid digits of $A_{00}^{(1)(1)}$ are not varied. The values of stress $\sigma_z^{(0)}$ (t_{max}) at points A and B (Fig. 1) for $X_{12} = 2.1p_x$, $2.2p_x$ and $2.5p_x$ are presented in Table 2. As expected, the series for stresses converges more slowly, especially when inhomogeneities are nearly placed. However, in this case also for $t_{max} = 15$ the deviation of $\sigma_z^{(0)}$ from the limit value does not exceed 1% at the point of maximum concentration (point B). For other points on the surface $\xi_1 = \xi_1^0$ and when X_{12} increases the convergence is more rapid. The numerical results presented below were calculated for $t_{max} = 15$ and $I_{max} = 25$.

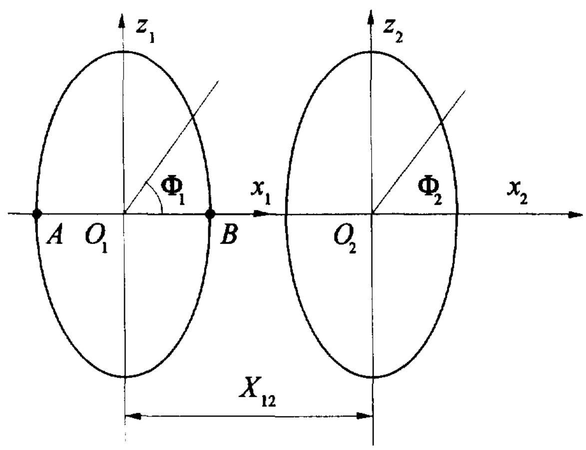

Fig. 1. Coordinates in the two-particle test problem.

Table 1. Convergence of $A_{00}^{(1)(1)}$ (t_{max}, l_{max}) for parameter values $\mu_1 = \mu_2 = 0$, $X_{12} = 2.1p_x$, $\varepsilon_{33}^{\infty} = \delta$ (uniaxial deformation along the Oz axis)

l_{max}	t_{max} = 1	3	5	7	9	11
1	5.691	5.691	5.691	5.691	5.691	5.691
3	5.839	5.956	5.984	5.990	5.990	5.990
5	5.837	5.932	5.953	5.960	5.963	5.963
7	5.837	5.929	5.944	5.949	5.952	5.952
9	5.837	5.929	5.940	5.945	5.947	5.948
11	5.837	5.929	5.940	5.943	5.944	5.945
13	5.837	5.929	5.940	5.942	5.943	5.943
15	5.837	5.929	5.940	5.941	5.942	5.942
17	5.837	5.929	5.940	5.941	5.942	5.942
19	5.837	5.929	5.940	5.941	5.941	5.942

Table 2. Convergence of stress $\sigma_z^{(0)}(t_{max})$ at points A ($\Phi_1 = \pi$) and B ($\Phi_1 = 0$) on the first cavity surface for $X_{12} = 2.1p_x$, $2.2p_x$ and $2.5p_x$ ($\mu_1 = \mu_2 = 0$)

	$X_{12} = 2.1p_x$		$X_{12} = 2.2p_x$		$X_{12} = 2.5p_x$	
t_{max}	B	A	B	A	B	A
1	4.96	4.96	4.91	4.91	4.81	4.81
3	6.29	4.98	5.97	4.89	5.40	4.67
5	7.35	4.86	6.66	4.77	5.77	4.65
7	7.97	4.77	6.98	4.71	5.80	4.64
9	8.32	4.73	7.12	4.68	5.80	4.64
11	8.51	4.70	7.18	4.67	5.80	4.64
13	8.63	4.69	7.21	4.67	5.80	4.64
15	8.68	4.69	7.22	4.67	5.80	4.64
17	8.70	4.69	7.22	4.67	5.80	4.64

The curves in Fig. 2 show the stress distribution $\sigma_z^{(0)}/\varepsilon_{33}^{\infty}$ on the surface of the first cavity ($\varphi_1 = 0$, $0 \leqslant \Phi_1 \leqslant \pi$) for $\mu_1 = \mu_2 = 0$, $\varepsilon_{33}^{\infty} = \delta$. In all pictures the dashed curve 1 corresponds to the solution for $X_{12} = \infty$ (space with a single cavity), the solid curves 2 and 3 are calculated for $X_{12} = 2.5p_x$ and $X_{12} = 2.1p_x$, respectively. The stress concentration $\sigma_z^{(0)}$ in the neighbourhood of point B increases when cavities are brought together; so, for $X_{12} = 2.1p_x$ it exceeds almost twice the stress at this point for a single cavity. At the same

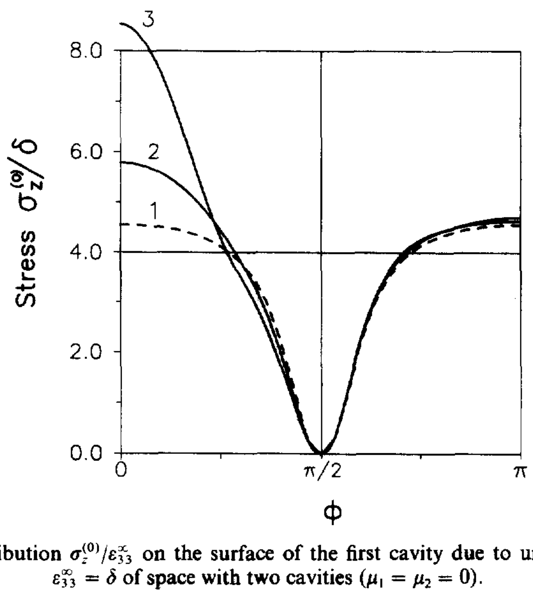

Fig. 2. Stress distribution $\sigma_z^{(0)}/\varepsilon_{33}^{\infty}$ on the surface of the first cavity due to uniaxial deformation $\varepsilon_{33}^{\infty} = \delta$ of space with two cavities ($\mu_1 = \mu_2 = 0$).

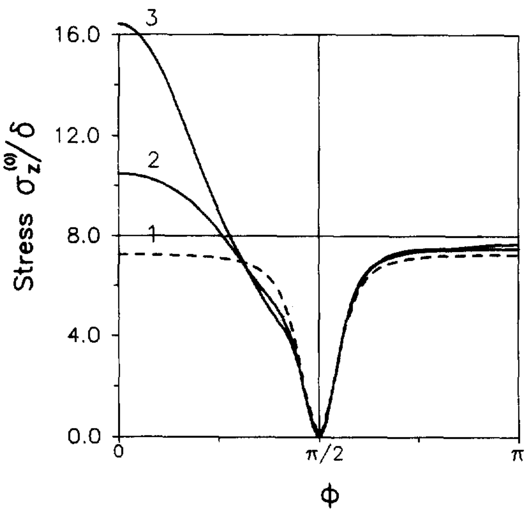

Fig. 3. Stress distribution $\sigma_z^{(0)}/\varepsilon_{33}^\infty$ on the surface of the first cavity due to all-round deformation $\varepsilon_{ii}^\infty = \delta$ of space with two cavities ($\mu_1 = \mu_2 = 0$).

time, the stress distribution on the opposite side of the spheroid (point A) is practically independent of X_{12}. The dependencies $\sigma_z^{(0)}(\Phi_1)$ also show similar behaviour due to the all-round deformation $\varepsilon_{ii}^\infty = \delta$ plotted in Fig. 3. They are calculated for the same parameters as in the previous case.

The next two pictures concern the medium with two inclusions, $\mu_1 = \mu_2 = 10$. The curves in Fig. 4 represent the matrix stress distribution $\sigma_z^{(0)}$ at the interface $\xi_1 = \xi_1^0$, $\varepsilon_{ii}^\infty = \delta$. The analogous curves for $\sigma_x^{(0)}$ are plotted in Fig. 5. Similarly to the case of cavities, the stress concentration in a zone between inclusions grows significantly when X_{12} decreases. So, $\sigma_x^{(0)}$ at the point B for $X_{12} = 2.1p_x$ (curve 3) exceeds more than twice the value calculated for $X_{12} = \infty$ (curve 1). The analogy with the case of cavities also becomes apparent in the localization of disturbance caused by interaction on the half-surface $-\pi/2 \leqslant \Phi_1 \leqslant \pi/2$. The stresses on the opposite side of the inclusion are only slightly influenced by the presence of a second inclusion.

The investigation of the interaction between hard inclusion and cavity is of interest because this situation is typical for many composites. So, the stress $\sigma_x^{(0)}$ (Φ_1) distribution on $\xi_1 = \xi_1^0$ (Fig. 6) corresponds to parameter values $\mu_1 = 10$, $\mu_2 = 0$ and $\varepsilon_{11}^\infty = \delta$ (deformation along the Ox axis). It can be seen that the neighbouring cavity decreases $\sigma_x^{(0)}$ at the

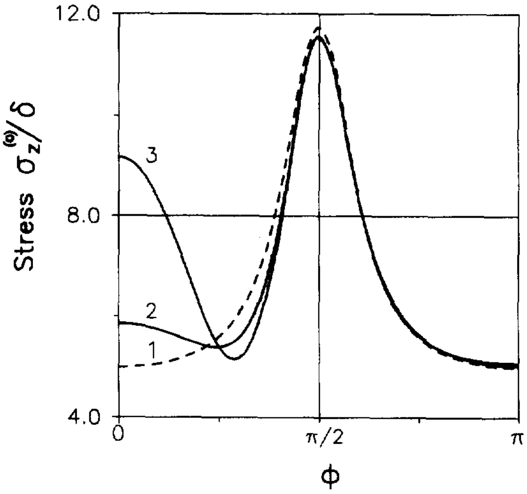

Fig. 4. Stress distribution $\sigma_z^{(0)}/\varepsilon_{33}^\infty$ on the interface $\xi_1 = \xi_1^0$ due to all-round deformation $\varepsilon_{ii}^\infty = \delta$ of space with two inclusions ($\mu_1 = \mu_2 = 10$).

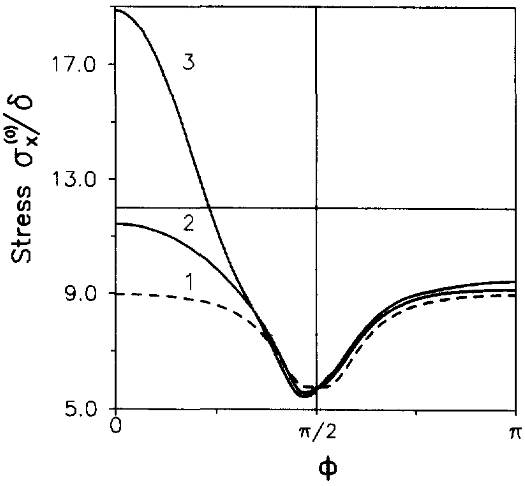

Fig. 5. Stress distribution $\sigma_x^{(0)}/\varepsilon_{33}^{\infty}$ on the interface $\xi_1 = \xi_1^0$ due to all-round deformation $\varepsilon_{ii}^{\infty} = \delta$ of space with two inclusions ($\mu_1 = \mu_2 = 10$).

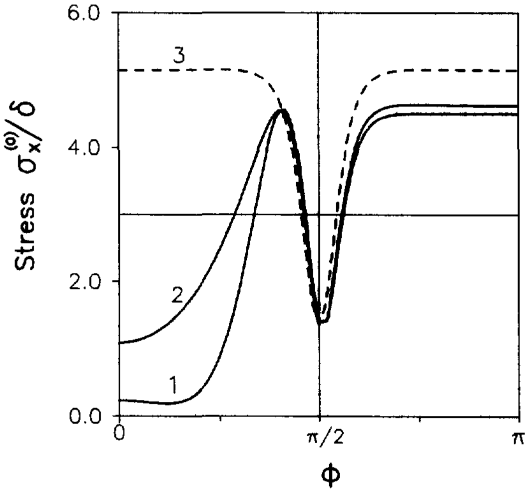

Fig. 6. Stress distribution $\sigma_x^{(0)}/\varepsilon_{33}^{\infty}$ on the interface $\xi_1 = \xi_1^0$ due to uniaxial deformation $\varepsilon_{11}^{\infty} = \delta$ of space with inclusion and cavity ($\mu_1 = 10$, $\mu_2 = 0$).

point B to almost zero, whereas for a single inclusion the stress at this point has a maximum. The stresses for $\Phi_1 > \pi/2$ also decrease but not so significantly.

The problem for a multi-particle model, when instead of strain tensor ε^{∞} the stress tensor σ^{∞} is given, can be examined in the same manner. Table 3 contains the values of $\sigma_z^{(0)}$ (Φ_1) on the cavity equator ($\Phi_1 = 0$ and $\Phi_1 = \pi$) due to uniaxial tension along the Oz axis $\sigma_{33}^{\infty} = 1$ of a space with two cavities ($\mu_1 = \mu_2 = 0$). The analogous results for $\sigma_y^{(0)}$ (Φ_1)

Table 3. Stress $\sigma_{z_{\max}}^{(0)}$ dependence upon distance between cavities due to uniaxial tension $\sigma_{33}^{\infty} = 1$ of a space with two cavities ($\mu_1 = \mu_2 = 0$)

	$p_z/p_x = 1.2$		$p_z/p_x = 2.0$	
$\frac{X_{12}}{p_x}$	$\Phi_1 = 0$	$\Phi_1 = \pi$	$\Phi_1 = 0$	$\Phi_1 = \pi$
∞	1.84	1.84	1.44	1.44
3.0	1.85	1.85	1.50	1.45
2.5	2.04	1.86	1.68	1.46
2.3	2.32	1.87	1.87	1.46
2.1	3.18	1.88	3.27	1.47

Table 4. Stress $\sigma^{(0)}_{y_{max}}$ dependence upon distance between cavities due to uniaxial tension $\sigma^{\infty}_{22} = 1$ of a space with two cavities ($\mu_1 = \mu_2 = 0$)

	$p_z/p_x = 1.2$		$p_z/p_x = 2.0$	
$\frac{X_{12}}{p_x}$	$\Phi_1 = 0$	$\Phi_1 = \pi$	$\Phi_1 = 0$	$\Phi_1 = \pi$
∞	2.17	2.17	2.48	2.48
3.0	2.18	2.18	2.52	2.51
2.5	2.39	2.20	2.88	2.54
2.3	2.75	2.21	3.43	2.56
2.1	3.94	2.23	5.22	2.61

Table 5. Stress $\sigma^{(0)}_{x_{max}}$ dependence upon distance between inclusions due to uniaxial tension $\sigma^{\infty}_{11} = 1$ of a space with two inclusions ($\mu_1 = \mu_2 = 100$)

	$p_z/p_x = 1.2$		$p_z/p_x = 2.0$	
$\frac{X_{12}}{p_x}$	$\Phi_1 = 0$	$\Phi_1 = \pi$	$\Phi_1 = 0$	$\Phi_1 = \pi$
∞	1.82	1.82	1.64	1.64
3.0	2.60	1.95	2.52	1.77
2.5	3.85	2.02	3.67	1.83
2.3	5.40	2.06	5.05	1.87
2.1	10.71	2.21	9.38	1.98

due to load $\sigma^{\infty}_{22} = 1$ are given in Table 4. The analysis of these data shows a significant stress concentration in area between closely placed pores. Even greater stress concentration arises among hard ($\mu_1 = \mu_2 = 100$) nearly-touching inclusions (Table 5). For instance, the coefficient of stress concentration $\sigma^{(0)}_x$ induced by uniaxial tension along the Ox axis ($X_{12} = 2.1p_x$) is equal to 5.87 for $p_z/p_x = 1.2$ and 4.74 for $p_z/p_x = 2.0$. To estimate the macroscopic properties (in particular, brittle strength) of composites properly, these effects must be taken into account.

5. CONCLUSIONS

The proposed rigorous method to solve the boundary-value problems of elasticity in a multiply-connected domain with spheroidal boundaries is simple and effective from a computational standpoint. It provides a high accuracy of solution and can be used to analyse a variety of multi-particle model problems of composite mechanics. In this way both disordered and periodic structures can be studied with a full account of interaction effects. In the same manner the problems for finite multiply-connected domains constrained by spheroidal boundaries, e.g. the double-inclusion model (Hori and Nemat-Nasser, 1993), can be considered. In this case one must also use the formulae for re-expansion of internal solutions through internal ones and of external solutions through external ones (Kushch, 1995).

In this paper we discussed only the case of aligned spheroidal inclusions. The case of arbitrary oriented inhomogeneities will be treated in a subsequent paper.

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APPENDIX A. VECTORIAL PARTIAL SOLUTIONS OF LAME'S EQUATION IN A SPHEROIDAL BASIS

The internal solutions (constrained at $\|\mathbf{r}\| \to \mathbf{0}$) are

$$\begin{aligned}
\mathbf{s}_{ts}^{(1)} &= \mathbf{e}_1 f_{t-1}^{s-1} - \mathbf{e}_2 f_{t-1}^{s+1} + \mathbf{e}_3 f_{t-1}^{s}\,;\\
\mathbf{s}_{ts}^{(2)} &= [\mathbf{e}_1 (t-s+1) f_t^{s-1} + \mathbf{e}_2 (t+s+1) f_t^{s+1} - \mathbf{e}_3 s f_t^s]/(t+1)\,;\\
\mathbf{s}_{ts}^{(3)} &= \mathbf{e}_1[-(x-iy)D_2 f_{t+1}^{s-1} - ((\xi^0)^2-1)D_1 f_t^s + (t-s+1)(t-s+2)\beta_t f_{t+1}^{s-1}]\\
&\quad + \mathbf{e}_2[(x+iy)D_1 f_{t+1}^{s+1} - ((\xi^0)^2-1)D_2 f_t^s - (t+s+1)(t+s+2)\beta_t f_{t+1}^{s+1}]\\
&\quad + \mathbf{e}_3[zD_3 f_{t+1}^s - (\xi^0)^2 D_3 f_t^s - C_{ts} f_{t+1}^s]\,;\\
t &= 0, 1, \ldots, \quad |s| \leqslant t; \quad \beta_t = \frac{t+5-4\nu}{(t+1)(2t+3)}, \quad C_{ts} = (t+s+1)(t-s+1)\beta_t.
\end{aligned} \tag{A1}$$

In eqns (A1), the following notations are adopted:

$$\mathbf{e}_1 = (\mathbf{e}_x + i\mathbf{e}_y)/2, \quad \mathbf{e}_2 = (\mathbf{e}_x - i\mathbf{e}_y)/2, \quad \mathbf{e}_3 = \mathbf{e}_z\,; \quad D_1 = (\partial/\partial x - i\partial/\partial y), \quad D_2 = (\partial/\partial x + i\partial/\partial y), \quad D_3 = \partial/\partial z, \tag{A2}$$

where $f_t^s = [(t-s)!/(t+s)!]\,P_t^s(\xi)\chi_t^s(\eta,\varphi)$ are the internal partial solutions of Laplace's equation in spheroidal coordinates (f, ξ, η, φ) defined as

$$x + iy = f\bar{\xi}\bar{\eta}\exp(i\varphi), \quad z = f\xi\eta, \quad \bar{\xi}^2 = \xi^2 - 1, \quad \bar{\eta}^2 = 1 - \eta^2\,; \quad \xi > 1, \quad |\eta| \leqslant 1, \quad 0 \leqslant \varphi \leqslant 2\pi; \tag{A3}$$

$\chi_t^s(\eta,\varphi) = P_t^s(\eta)\exp(is\varphi)$ are the scalar spherical harmonics and P_t^s are the associated Legendre functions of first kind. The equalities (A3) at $\mathrm{Re}(f) > 0$ describe the family of confocal prolate spheroids with inter-foci distance $2f$. The solutions introduced herein are written in the prolate-spheroidal basis. In the case of an oblate spheroid one must replace ξ on $i\bar{\xi}$ and f on $(-if)$ in eqn (A1) and all the following formulae.

The external solutions (constrained at $\|\mathbf{r}\| \to \infty$) are:

$$\begin{aligned}
\mathbf{S}_{ts}^{(1)} &= \mathbf{e}_1 F_{t+1}^{s-1} - \mathbf{e}_2 F_{t+1}^{s+1} + \mathbf{e}_3 F_{t+1}^{s}\,;\\
\mathbf{S}_{ts}^{(2)} &= [\mathbf{e}_1 (t+s) F_t^{s-1} + \mathbf{e}_2 (t-s) F_t^{s+1} + \mathbf{e}_3 s F_t^s]/t\,;\\
\mathbf{S}_{ts}^{(3)} &= \mathbf{e}_1[-(x-iy)D_2 F_{t-1}^{s-1} - ((\xi^0)^2-1)D_1 F_t^s + (t+s)(t+s-1)\beta_{-(t+1)} F_{t-1}^{s-1}]\\
&\quad + \mathbf{e}_2[(x+iy)D_1 F_{t-1}^{s+1} - ((\xi^0)^2-1)D_2 F_t^s - (t-s)(t-s-1)\beta_{-(t+1)} F_{t-1}^{s+1}]\\
&\quad + \mathbf{e}_3[zD_3 F_{t-1}^s - (\xi^0)^2 D_3 F_t^s - C_{-(t+1),s} F_{t-1}^s]\,;\\
t &= 0, 1, \ldots; \quad |s| \leqslant t,
\end{aligned} \tag{A4}$$

where $F_t^s = [(t-s)!/(t+s)!]\,Q_t^s(\xi)\chi_t^s(\eta,\varphi)$ are the external solutions of Laplace's equation in spheroidal coordinates and Q_t^s are the associated Legendre functions of the second kind. It is easy to see that eqns (A4) can be obtained from eqns (A1) by replacing the index t with $-(t+1)$ and $f_{-(t+1)}^s$ with F_t^s. The functions (A1) and (A4) are sufficient to solve in series the boundary-value problems for a single-connected body with a spheroidal boundary.

To satisfy the boundary conditions on the spheroidal surface $\xi = \xi^0$ the components of the displacement vector must be expanded over scalar harmonics χ_t^s. For external partial solutions such expansions have the form

$$\mathbf{S}_{ts}^{(1)} = \mathbf{e}_1 \frac{(t-s+2)!}{(t+s)!} Q_{t+1}^{s-1} \chi_{t+1}^{s-1} - \mathbf{e}_2 \frac{(t-s)!}{(t+s+2)!} Q_{t+1}^{s+1} \chi_{t+1}^{s+1} + \mathbf{e}_3 \frac{(t-s+1)!}{(t+s+1)!} Q_{t+1}^{s} \chi_{t+1}^{s};$$

$$\mathbf{S}_{ts}^{(2)} = \mathbf{e}_1 \frac{(t-s+1)!}{(t+s-1)!} \frac{(t+s)}{t} Q_t^{s-1} \chi_t^{s-1} + \mathbf{e}_2 \frac{(t-s-1)!}{(t+s+1)!} \frac{(t-s)}{t} Q_t^{s+1} \chi_t^{s+1}$$
$$+ \mathbf{e}_3 \frac{(t-s)!}{(t+s)!} \frac{s}{t} Q_t^s \chi_t^s;$$

$$\mathbf{S}_{ts}^{(3)} = \mathbf{e}_1 \frac{(t-s)!}{(t+s-2)!} \{-(t-s+1)\xi Q_t^{s-1} + (t+s-1)[1+(t+s)\beta_{-(t+1)}] Q_{t-1}^{s-1}\} \chi_{t-1}^{s-1}$$
$$+ \mathbf{e}_2 \frac{(t-s-2)!}{(t+s)!} \{(t-s-1)\xi Q_t^{s+1} - (t-s-1)[1+(t-s)\beta_{-(t+1)}] Q_{t-1}^{s+1} \chi_{t-1}^{s+1}$$
$$- \mathbf{e}_3 \frac{(t-s-1)!}{(t+s-1)!} \{-(t-s)\xi Q_t^s - C_{-(t+1),s} Q_{t-1}^s\} \chi_{t-1}^s. \qquad \text{(A5)}$$

For internal solutions the expressions are quite similar.

APPENDIX B. ADDITION THEOREMS FOR EXTERNAL SOLUTIONS OF LAME'S EQUATION IN A SPHEROIDAL BASIS

Let $O_i x_i y_i z_i$ $(i = 1, 2)$ be two identically oriented coordinate systems; their radius vectors satisfy the condition $\mathbf{r}_1 = \mathbf{R}_{12} + \mathbf{r}_2$. Then

$$\mathbf{S}_{ts}^{(i)}(\mathbf{r}_1, f_1) = \sum_{j=1}^{3} \sum_{k=0}^{\infty} \sum_{l=-k}^{k} \eta_{tksl}^{(i)(j)}(\mathbf{R}_{12}, f_1, f_2) \mathbf{s}_{kl}^{(j)}(\mathbf{r}_2, f_2), \quad i = 1,2,3; \quad t = 0,1,\ldots; \quad |s| \leqslant t, \qquad \text{(B1)}$$

where

$$\eta_{tksl}^{(1)(2)} = \eta_{tksl}^{(1)(3)} = \eta_{tksl}^{(2)(3)} = 0; \quad \eta_{tksl}^{(1)(1)} = \eta_{t+1,k-1}^{s-l};$$
$$\eta_{tksl}^{(2)(1)} = \left(\frac{s}{t} + \frac{l}{k}\right) \eta_{t,k-1}^{s-l}, \quad k > 0; \quad \eta_{t0s0}^{(2)(1)} = 0;$$
$$\eta_{tksl}^{(3)(3)} = \eta_{t-1,k+1}^{s-l};$$
$$\eta_{tksl}^{(3)(2)} = 4(1-\nu)\left(\frac{s}{t} + \frac{l}{k}\right) \eta_{t,k-1}^{s-l}, \quad k > 0; \quad \eta_{t0s0}^{(3)(2)} = 0;$$
$$\eta_{tksl}^{(3)(1)} = 4(1-\nu)\frac{l}{k}\left(\frac{s}{t} + \frac{l}{(k-1)}\right) + C_{k-2,l} - C_{-(t+1),s} \eta_{t-1,k-1}^{s-l}$$
$$+ (2k-1) \sum_{p=0}^{\infty} \left[\frac{Z_{12}}{f_1} \eta_{t-1,k+2p}^{s-l} + f_1 (\xi_1^0)^2 \eta_{t-1,k+2p+1}^{s-l} - f_2 (\xi_2^0)^2 \eta_{t,k+2p}^{s-l}\right], \quad k \geqslant 2;$$
$$\eta_{t1s0}^{(3)(1)} = -C_{-(t+1),s} \eta_{t-1,0}^{s},$$
$$\eta_{t1s1}^{(3)(1)} = (t+s-1)[1+(t+s)\beta_{-(t+1)}] \eta_{t-1,0}^{s-1},$$
$$A_{t1s,-1}^{(3)(1)} = (t-s-1)[1+(t-s)\beta_{-(t+1)}] \eta_{t-1,0}^{s+1}. \qquad \text{(B2)}$$

In eqns (B2), η_{tk}^{s-l} are the expansion coefficients of the external solution of Laplace's equation:

$$F_t^s(\mathbf{r}_1, f_1) = \sum_{k=0}^{\infty} \sum_{l=-k}^{k} \eta_{tk}^{s-l}(\mathbf{R}_{12}, f_1, f_2) f_k^l(\mathbf{r}_2, f_2), \qquad \text{(B3)}$$

$$\eta_{tk}^{s-l} = a_{tk} \left(\frac{2}{\bar{f}}\right)^{t+k+1} \sum_{r=0}^{\infty} F_{t+k+2r}^{s-l}(\mathbf{R}_{12}, \bar{f}) \sum_{n=0}^{r} \frac{(-1)^{r-n}}{(r-n)!} \left(\frac{f_1}{\bar{f}}\right)^{2n}$$
$$\times (t+k+2r-2n+1/2)\Gamma(t+k+r+n+1/2) M_{tkn}(f_1, f_2),$$
$$a_{tk} = (-1)^k (k+1/2)\sqrt{\pi} \left(\frac{f_1}{2}\right)^{t+1} \left(\frac{f_2}{2}\right)^k,$$
$$M_{tkn}(f_1, f_2) = \sum_{j=0}^{n} \frac{(f_2/f_1)^{2j}}{j!(n-j)!\Gamma(t+n-j+3/2)\Gamma(k+n+3/2)}, \qquad \text{(B4)}$$

$\bar{f} > f_1$; $\Gamma(z)$ is the gamma function. The series (B3) as well as (B1) converges for the case of two prolate spheroids in a region constrained by the spheroid $\xi_2 = \xi^0$ with inter-foci distance $2f_2$ and with centre at the point O_2 if the point O_1 lies outside the spheroid with semi-axes $d_2\bar{\xi}^0$ and $(f_2\xi^0 + f_1)$. The geometrical sense of the condition consists of non-intersection of the spheroid $\xi_2 = \xi^0$ and the infinitely thin spheroid with interfacial distance $2f_1$ centred at point O_1. For other cases (two oblate spheroids, prolate and oblate ones) the convergency condition has an analogous sense. This provides the applicability of these results to solve the problems for a multiply-connected domain with non-touching spheroidal boundaries. Note that a simpler expression exists for η_{tk}^{s-l}. It has the form

$$\eta_{tk}^{s-l} = a_{tk}\sqrt{\pi}\sum_{r=0}^{\infty}\left(\frac{f_1}{2}\right)^{2r} M_{tkr}(f_1,f_2)\, Y_{t+k+2r}^{s-l}(\mathbf{R}_{12}), \tag{B5}$$

where

$$Y_t^s = \frac{(t-s)!}{r^{t+1}} P_t^s(\cos\theta)\exp(is\varphi).$$

The convergency condition of series (B5) is $\|R_{12}\| < 2f_1$. Hence, this expression can be used when the indicated geometrical inequality is satisfied. For $|s-l| = t+k$ we must also define $\eta_{t-1,k-1}^{s-l}$. So, for $\|\mathbf{R}_{12}\| > 2f_1$, η_{tk}^{t+k+2} has the form (B5), where at $s = t+2$ Y_t^s must be replaced by $(x+iy)Y_{t+1}^{t+1}$. In a general case

$$\eta_{tk}^{t+k+2} = -a_{tk}\left(\frac{2}{\tilde{f}}\right)^{t+k+1}\Bigg[\sum_{r=0}^{\infty} F_{t+k+2r+2}^{t+k+2}(\mathbf{R}_{12},\tilde{f})\sum_{l=0}^{r}\frac{(-1)^{r-l}}{(r-l+1)!}\left(\frac{f_1}{\tilde{f}}\right)^{2l}$$
$$\times(t+k+2r-2l+5/2)\Gamma(t+k+r+l+3/2)M_{tkl}(f_1,f_2)+\frac{2}{f_1}(X_{12}+iY_{12})M_{tk0}$$
$$\times\sum_{r=0}^{\infty}F_{t+k+2r+1}^{t+k+1}(\mathbf{R}_{12},\tilde{f})\frac{(-1)^r}{r!}(t+k+2r+3/2)\Gamma(t+k+r+3/2)\Bigg]. \tag{B6}$$

It should be noted that calculation of the sum on p in the expression for $\eta_{tksl}^{(3)(1)}$ is not a problem; by replacing the summation order it reduces to a form similar to eqns (B4) and (B5).